

Old and new algorithms for π

This letter concerns Semjon Adlaj's article *An eloquent formula for the perimeter of an ellipse* [Notices 59, 8 (Sept. 2012), 1094–1099]. In his comments on the “(so-called) Brent-Salamin algorithm” for computing π , Prof. Adlaj misses some important points.

First, both Brent and Salamin acknowledged their debt to Gauss and Legendre. That the names “Brent-Salamin” or “Salamin-Brent” are widely used is probably due to the ambiguity of calling something new after Gauss and Legendre, e.g. a Google search for “Gauss-Legendre” gives many hits on Gauss-Legendre quadrature.

Second, although Euler discovered the special case of Legendre's relation that is used in the simplest Brent-Salamin algorithm ($k = k' = 1/\sqrt{2}$), the more general form of Legendre's relation is needed for the members of the family of algorithms that arise from choosing $k \neq k'$. Since Legendre's relation is not attributed to Euler, it would be uninformative to use the name “Gauss-Euler” as Prof. Adlaj suggests [footnote 4]. A Google search for “Gauss-Euler” gives even more hits than one for “Gauss-Legendre”, but they are almost all irrelevant.

Third, and more important, none of those three great mathematicians of the past would have appreciated the significance of such an algorithm, because they lived in the days before electronic computers and fast algorithms, such as the Schönhage-Strassen algorithm, for multiplication of large integers. Without such technology and modern algorithms, the Brent-Salamin algorithm is a relatively poor algorithm for computing π – algorithms based on the Maclaurin series for $\arctan(1/n)$, such as Machin's $\pi/4 = 4 \arctan(1/5) - \arctan(1/239)$, are far superior (even today, they are competitive if combined with binary splitting and fast multiplication algorithms). Indeed, on reading Gauss's unpublished notebook entry of May 1809, it seems probable that he did not regard his discovery as an algorithm for computing π , since π only appears in the denominator of the right-hand side of the crucial equation. More likely Gauss regarded this equation as an interesting identity involving elliptic integrals, only incidentally involving the known constant π . [The relevant notebook entry is reproduced on page 99 of the book *Pi: Algorithms, Computer, Arithmetik* by Arndt and Haenel.]

Finally, perhaps this emphasis on the computation of a single constant is unwarranted. Brent's 1975 and 1976 papers, not referenced by Prof. Adlaj, showed that *all* elementary functions can be evaluated to given accuracy just as fast as π , up to a constant factor, by using the arithmetic-geometric mean. This of course includes the computation of an infinite set of constants such as e^π and π/e . No doubt this fact would have been of more interest to Euler, Legendre and Gauss than yet another formula or algorithm for π .

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